ON THE THETA OPERATOR FOR MODULAR FORMS MODULO PRIME POWERS

IMIN CHEN, IAN KIMING

Abstract. We consider the classical theta operator θ on modular forms modulo p^m and make a detailed study of its weight properties in the case of $m = 2$ and level 1. The detailed study is restricted to the case of level 1 where the situation can be made explicit using the classical approach of Ramanujan, Swinnerton-Dyer, and Serre. However, part of the conclusions generalize to arbitrary $m \geq 2$ and general level N prime to p.

1. Introduction

Let p be a prime number. We shall assume $p \geq 5$ throughout the paper in order to avoid certain technicalities when p is 2 or 3. Also let $N \in \mathbb{N}$ be prime to p.

Further let $k, m \in \mathbb{N}$, $k > 1$, denote by $M_k(N, \mathbb{Z}_p)$ the $\mathbb{Z}_p$-module of modular forms of weight k for $\Gamma_1(N)$ over $\mathbb{Z}_p$, and let $M_k(N, \mathbb{Z}/p^m\mathbb{Z})$ be the $\mathbb{Z}/p^m\mathbb{Z}$-module of modular forms for $\Gamma_1(N)$ over $\mathbb{Z}/p^m\mathbb{Z}$. Thus, $M_k(N, \mathbb{Z}/p^m\mathbb{Z})$ is the image under the natural reduction map $M_k(N, \mathbb{Z}_p) \to M_k(N, \mathbb{Z}/p^m\mathbb{Z})$. If $N = 1$ we shall suppress N from the notation, writing just $M_k(\mathbb{Z}_p)$ etc.

Let $k_1, \ldots, k_t$ be a collection of weights and let $f_i \in M_{k_i}(N, \mathbb{Z}_p)$. The q-expansion of an element in a direct sum of the $M_{k_i}(N, \mathbb{Z}_p)$'s or $M_{k_i}(N, \mathbb{Z}/p^m\mathbb{Z})$'s is defined by extending linearly on each component. When we write $f_1 + \ldots + f_t \equiv 0 \pmod{p^m}$, we shall mean that the q-expansion $f_1(q) + \ldots + f_t(q)$ lies in $p^m\mathbb{Z}_p[[q]]$. Similarly for $f_i \in M_{k_i}(N, \mathbb{Z}/p^m\mathbb{Z})$, we write $f_1 + \ldots + f_t \equiv 0 \pmod{p^m}$ if the q-expansion of $f_1 + \ldots + f_t$ equals 0 in $(\mathbb{Z}/p^m\mathbb{Z})[[q]]$. In such a case, we say that $f_1 + \ldots + f_t$ is congruent to 0 modulo p^m.

Let us recall the definition and basic properties of the standard Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$, cf. §1 of [9], for instance: the series

$$G_k := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

with B_k the k'th Bernoulli number and $\sigma_t(n) := \sum_{d|n} d^t$ the usual divisor sum, is (with $q := e^{2\pi i z}$) for k an even integer ≥ 4 a modular form on $\mathrm{SL}_2(\mathbb{Z})$. Defining E_k as the normalization

$$E_k := -\frac{2k}{B_k} \cdot G_k$$

one has $E_k \equiv 1 \ (p^t)$ when (and only when) $k \equiv 0 \ (p^{t-1}(p-1))$.

There are natural inclusions (preserving q-expansions)

$$M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \hookrightarrow M_{k+p^{m-1}(p-1)}(N, \mathbb{Z}/p^m\mathbb{Z}),$$

induced by multiplication by $E_{p-1}^{p^{m-1}}$, using the fact that $E_{p-1}^{p^{m-1}} \equiv 1 \pmod{p^m}$. Note that $E_{p^{m-1}(p-1)} = E_{p-1}^{p^{m-1}}$ in $M_k(N, \mathbb{Z}/p^m\mathbb{Z})$, again by the q-expansion principle, so that the map can also be seen as induced by multiplication by $E_{p^{m-1}(p-1)}$.

As is well-known, when we specialize the above series for G_k to $k = 2$ and define

$$G_2 := -\frac{B_2}{4} + \sum_{n=1}^{\infty} \sigma_1(n) q^n = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

then G_2 does not represent a modular form in the usual sense, but does so in the p-adic sense, cf. [9], §2. One defines E_2 as the normalization of G_2, i.e., $E_2 := -24G_2$. Thus, E_2 is also a p-adic modular form.

Consider the classical theta operator $\theta f = \frac{k}{12} E_2 + \frac{1}{12} \partial f$ of Ramanujan. Its effect on q-expansions is $\sum_n a_n q^n \mapsto \sum_n n a_n q^n$. Since E_2 is a p-adic modular form it is for any $m \in \mathbb{N}$ congruent modulo p^m to a classical modular form of some weight. Thus we have $E_2 \equiv E_{p+1} \pmod{p}$, for example, and since we classically know that ∂ maps modular forms of weight k to modular forms of weight $k+2$, one obtains the classical operator θ that maps $M_k(N, \mathbb{F}_p)$ to $M_{k+p+1}(N, \mathbb{F}_p)$. Studying this operator as well as its interaction with the 'weight filtration' (see below) is a key tool in the theory of modular forms modulo p; cf. for instance Jochnowitz' proof of finiteness of systems of Hecke eigenvalues mod p across all weights in [4], or Edixhoven's results on the optimal weight in Serre's conjectures [3].

As we have launched a general study of modular forms mod p^m in [2] and [1] it is natural to ask whether a θ operator with similar properties can be defined on such forms: one part of the motivation is obviously to investigate to whether Jochnowitz' arguments extend in any straightforward manner. But another part of the motivation is that consideration of the θ operator is very natural from the point of view of Galois representations: We have discussed in [1] how one attaches Galois representations to eigenforms mod p^m, and it is clear from the properties of those attached representations that applying the θ operator corresponds on the Galois side to twisting by the cyclotomic character mod p^m. Since this is a very natural operation on the Galois side, it is natural – and indeed necessary – to obtain a more detailed insight into what this operation means on the modular forms side.

Notice that Serre shows in [9, Théorème 5] that there exists a θ operator on p-adic modular forms (of level 1) whose effect on q-expansions is $\sum_n a_n q^n \mapsto \sum_n n a_n q^n$ and that sends a form of (p-adic) weight k to a form of weight $k+2$. One can view our results as finding a partially explicit expression with explicit weights for the mod p^m reduction of this operator.

Hence, our results are then that on the one hand an extension of the θ operator from the mod p to the mod p^m situation is indeed possible, but that the interplay of the θ operator with the weights of the forms becomes much more complicated when $m > 1$ and that, in fact, there are certain genuine qualitative differences between the case $m = 1$ and the general cases $m > 1$. Let us explain in detail.

We show that a θ operator on modular forms mod p^m can be defined such that θ maps $M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z})$ with $k(m) := 2 + 2p^{m-1}(p-1)$, such that the effect on q-expansions is $\sum_n a_n q^n \mapsto \sum_n n a_n q^n$, and such that θ satisfies simple commutation rules with Hecke operators T_ℓ for primes $\ell \neq p$. Cf. Theorem 1 below (and also Proposition 3 for a more detailed statement) as well as Proposition 5. The proofs use a number of results from [9] plus the observation that $f \mid V \equiv f^p \pmod{p}$ where V is the classical V operator.

If we define the weight $w_{p^m}(f)$ of an modular form f mod p^m, $f \not\equiv 0 \pmod{p}$, to be the smallest k such that f is congruent modulo p^m to an element of $M_k(N, \mathbb{Z}/p^m\mathbb{Z})$ then a classical fact, crucial for instance in the work [4], is that when $m = 1$ we have $w_p(\theta f) \leq w_p(f) + p + 1$ with equality if (and only if) $p \nmid w_p(f)$.

One might expect the generalization of this to be that $w_{p^m}(\theta f) \leq w_{p^m}(f) + 2 + p^{m-1}(p-1)$ and that one has equality in some cases. However, we shall show that this is false:

In section 3, we consider the case $N = 1$, $m = 2$. Here, the expression for θ can be made completely explicit and we analyze in this case the relation between $w_{p^2}(f)$ and $w_{p^2}(\theta f)$ in detail. We show in particular that there are cases where

$$w_{p^2}(\theta f) = w_{p^2}(f) + 2 + 2p^{m-1}(p-1) = w_{p^2}(f) + 2 + 2p(p-1) = w_{p^2}(f) + k(2),$$

where $k(m) = 2 + 2p^{m-1}(p-1)$, cf. Theorem 5 below. For example we have the following immediate corollary to Theorem 5:

Corollary 1. *Let $f \in M_k(\mathbb{Z}/p^2\mathbb{Z})$, $f \not\equiv 0 \pmod{p}$, and suppose that $w_{p^2}(f) = w_p(f) = k$ and that $p \nmid k$.*

Then $w_{p^2}(\theta f) = k + 2p(p-1) + 2$.

As a matter of fact, this particular corollary has a fairly immediate generalization to the case of arbitrary N (prime to p) and arbitrary $m \geq 2$: If $m \geq 2$, $f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z})$ with $f \not\equiv 0 \pmod{p}$, and if $w_{p^m}(f) = w_{p^{m-1}}(f) = \ldots = w_p(f) = k$ where $p \nmid k$ then

$$w_{p^m}(\theta f) = k + 2p^{m-1}(p-1) + 2.$$

The proof of this fact follows along the same lines as in the proof of Theorem 5, but uses the weight filtration results of [6], which are applicable to general level N (prime to p). If we consider the Eisenstein series E_{p-1} and E_{p+1} as modular forms modulo p (in the sense of Katz) then E_{p-1} (Hasse invariant) has only simple zeros and no zero common with E_{p+1}; cf. [6, Remark on p. 57]. This allows one to argue as in Theorem 5 about the weight filtration of the last term of the expression for the θ modulo p^m, which is the controlling term. We are indebted to Nadim Rustom for this observation, whose continuing work on these topics will appear elsewhere.

However, as Theorem 5 shows, when $m = 2$ the relation between $w_{p^2}(\theta f)$ and $w_{p^2}(f)$ appears far more complicated that for the case $m = 1$, and the more delicate parts of the Theorem 5 would seem to require more work to generalize as the proof relies on an explicit expression for θ modulo p^2.

The proofs in section 3 are via the explicit methods of Swinnerton-Dyer and Serre [8]. The reader should notice that for simplicity we have chosen to formulate the results for modular forms with coefficients in $\mathbb{Z}_p$ and hence reductions with coefficients in $\mathbb{Z}/p^m\mathbb{Z}$. However, the results carry over immediately without change to more general coefficients via base change (see e.g. [1], section 2.4).

2. The theta operator modulo prime powers

2.1. **Eisenstein series.** We shall now develop an explicit expression for the truncation modulo p^m of the p-adic Eisenstein series G_2.

Proposition 1. *Let $m \in \mathbb{N}$. Define the positive even integers $k_0, \ldots, k_{m-1}$ as follows: If $m \geq 2$, define:*

$$k_j := 2 + p^{m-j-1}(p^{j+1} - 1) \quad \textit{for } j = 0, \ldots, m-2$$

and

$$k_{m-1} := p^{m-1}(p+1)$$

and define just $k_0 := p+1$ *if* $m=1$.

Then $k_0 < \ldots < k_{m-1}$ *and there are modular forms* $f_0, \ldots, f_{m-1}$, *depending only on* p *and* m, *of weights* $k_0, \ldots, k_{m-1}$, *respectively, that have rational* q*-expansions, satisfy* $v_p(f_j) = 0$ *for all* j, *and are such that*

$$G_2 \equiv \sum_{j=0}^{m-1} p^j f_j \pmod{p^m}$$

as a congruence between (formal) q*-expansions.*

When $m = 2$ we can, and will, be a bit more explicit:

Proposition 2. *We have:*

$$G_2 \equiv f_0 + p \cdot f_1 \pmod{p^2}$$

with modular forms f_0 *and* f_1 *of weights* $2 + p(p-1)$ *and* $p(p+1)$, *respectively, explicitly:*

$$G_2 \equiv G_{2+p(p-1)} + p \cdot G_{p+1}^p \pmod{p^2}.$$

It is amusing to note the following consequence of the Proposition: For $p \neq 2, 3$, we have the following congruence of Bernoulli numbers,

$$\frac{B_2}{2} \equiv \frac{B_{p(p-1)+2}}{p(p-1)+2} + p\frac{B_{p+1}}{p+1} \pmod{p^2}.$$

However, this can also be seen in terms of p-adic continuity of Bernoulli numbers (cf. [10], Cor. 5.14 on p. 61, for instance).

Before the proofs of these propositions we need a couple of preparations.

Let k be an even integer ≥ 2. Recall from [9] that if we choose a sequence of even integers k_i such that $k_i \to \infty$ in the usual, real metric, but $k_i \to k$ in the p-adic metric, then the sequence G_{k_i} has a p-adic limit denoted by G_k^*. This series G_k^* is a p-adic modular form of weight k. It does not depend on the choice of the sequence k_i. In particular, we can, and will, choose $k_i := k + p^{i-1}(p-1)$, because if we chose another $k_i' = k + \lambda p^{i-1}(p-1)$, where $\lambda \geq 2$, then $G_{k_i'} \equiv G_{k_i} \pmod{p^i}$ so that $G_{k_i'} = G_{k_i} E_{p-1}^{p(\lambda-1)}$ in $M_{k_i'}(\mathbb{Z}/p^i\mathbb{Z})$ is not essentially different.

Lemma 1. *Let* k *be an even integer* ≥ 2 *and assume* $(p-1) \nmid k$. *Let* $t \in \mathbb{N}$.

Then $G_k^* \equiv G_{k+p^{t-1}(p-1)}$ (p^t).

Proof. Let $u, v \geq t$. We claim that $G_{k_u} \equiv G_{k_v}$ (p^t). Since the series G_{k_i} converges p-adically to G_k^*, the claim clearly implies the Lemma.

If $u = v$ the claim is trivial, so suppose not, say $u < v$. Then $k_v - k_u = p^{v-1}(p-1) - p^{u-1}(p-1)$ is a multiple of $p^{t-1}(p-1)$, say $k_v - k_u = s \cdot p^{t-1}(p-1)$. We also have $k_v - k_u \geq 4$. Hence, we find that $G := G_{k_u} \cdot E_{p^{t-1}(p-1)}^s$ is a modular form of weight k_v, and we have $G_{k_u} \equiv G$ (p^t).

Now notice that, when $i \geq t$, we have:

$$\sigma_{k_i-1}(n) = \sum_{d|n} d^{k-1+p^{i-1}(p-1)} \equiv \sum_{\substack{d|n \\ p\nmid d}} d^{k-1} \quad (p^t)$$

as $d^{p^{i-1}(p-1)} \equiv 1 \ (p^t)$ when $p \nmid d$ and $i \geq t$, and as $d^{p^{i-1}(p-1)} \equiv 0 \ (p^t)$ when $p \mid d$ and $i \geq t$ (because $p^{t-1}(p-1) \geq t$ as long as $p \neq 2$).

We conclude that the nonconstant terms of the series G_{k_u} and G_{k_v} are termwise congruent modulo p^t. The same is then true of the forms G and G_{k_v} that are both forms of weight k_v. Hence, the nonconstant terms of the form $(G - G_{k_v})/p^t$ are all p-integral. As $k_v \equiv k \not\equiv 0 \pmod{(p-1)}$, it follows from Théorème 8 of [8] that the constant term of this form is in fact also p-integral. Hence,

$$G_{k_u} \equiv G \equiv G_{k_v} \quad (p^t)$$

as desired. □

Recall that the V operator is defined on formal q-expansions as

$$\left(\sum a_n q^n\right) \mid V := \sum a_n q^{np}.$$

Corollary 2. *We have*

$$G_2 \equiv \sum_{j=0}^{m-1} p^j \cdot \left(G_{2+p^{m-j-1}(p-1)} \mid V^j\right) \pmod{p^m}$$

as a congruence between formal q-expansions.

Proof. Recall from [9], §2, the identity, valid for any even integer $k \geq 2$, that

$$G_k = G_k^* + p^{k-1}\left(G_k^* \mid V\right) + \ldots + p^{t(k-1)}\left(G_k^* \mid V^t\right) + \ldots.$$

The identity is first an identity of formal q-expansions, but then shows that G_k is a p-adic modular form as V acts on p-adic modular forms, cf. [9], §2.

If we specialize this identity to the case $k = 2$, reduce modulo p^m, and note that the previous Lemma applies since $(p-1) \nmid 2$, the claim follows immediately. □

We can also see the V operator as an operator on modular forms: Suppose that $f \in M_k(\mathrm{SL}_2(\mathbb{Z}); \mathbb{C})$. Then $(f \mid V)(z) = f(pz)$, and as is well-known $f \mid V \in M_k(\Gamma_0(p); \mathbb{C})$. The proof of the next lemma is a simple application of section 3.2 of [9]. Recall that if $f \in M_k(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Q}_p)$ is nonzero with q-expansion $\sum_n a_n q^n$ we define $v_p(f) := \min\{v_p(a_n) \mid \ n \in \mathbb{N}\}$ where $v_p(a_n)$ is the usual (normalized) p-adic valuation of a_n.

Lemma 2. *Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Q})$ and suppose $v_p(f) = 0$. Let $t \in \mathbb{N}$ and suppose that $s \in \mathbb{Z}_{\geq 0}$ is such that*

$$\inf(s+1, p^s + 1 - k) \geq t.$$

Then there is $h \in M_{k+p^s(p-1)}(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Q})$ with $v_p(h) = 0$, and such that

$$f \mid V \equiv h \pmod{p^t}.$$

Proof. As we noted above, $f \mid V$ is a modular form of weight k on $\Gamma_0(p)$. Since $f \mid V = \sum a_n q^{np}$ if $f = \sum a_n q^n$ we have $v_p(f \mid V) = 0$. Recall the Fricke involution for modular forms on $\Gamma_0(p)$ given by the action of the matrix

$$W = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right).$$

Since

$$f \mid V = p^{-k/2} f \mid_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

(recall that the weight k action is normalized so that diagonal matrices act trivially), since

$$\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right),$$

and since f is on $\mathrm{SL}_2(\mathbb{Z})$ we see that $f \mid VW = p^{-k/2} f$ so that $v_p(f \mid VW) = -k/2$.

Now let $E := E_{p-1}$ and put

$$g := E - p^{p-1}(E \mid V)$$

so that g is a modular form of weight $p-1$ on $\Gamma_0(p)$. Then, if we put

$$f_s := \mathrm{tr}((f \mid V) \cdot g^{p^s})$$

for $s \in \mathbb{Z}_{\geq 0}$ where tr denotes the trace from $\Gamma_0(p)$ to $\mathrm{SL}_2(\mathbb{Z})$, it follows from section 3.2 of [9] that f_s is a modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $k + p^s(p-1)$ and rational q-expansion. Furthermore, Lemme 9 of [9] implies that

$$\begin{aligned} v_p(f_s - (f \mid V)) &\geq \inf(s+1, p^s + 1 + v_p(f \mid VW) - k/2) \\ &= \inf(s+1, p^s + 1 - k) \geq t. \end{aligned}$$

Hence, we can choose $h := f_s$. As $f \mid V \equiv h \pmod{p^t}$ and $v_p(f \mid V) = 0$, we must have $v_p(h) = 0$. □

Proof of Proposition 1: That the defined weights $k_0, \ldots, k_{m-1}$ satisfy $k_0 < \ldots < k_{m-1}$ is verified immediately.

Thus, starting with Corollary 2 we see that it suffices to show that for each $j \in \{0, \ldots, m-1\}$ there is a modular form f_j of weight k_j with rational q-expansion and $v_p(f_j) = 0$, and such that

$$G_{2+p^{m-j-1}(p-1)} \mid V^j \equiv f_j \pmod{p^{m-j}}.$$

If $m = 1$, $j = 0$ we just take $f_0 := G_{p+1}$, so assume $m \geq 2$. Then, if $j = m-1$, note that

$$G_{p+1} \mid V^{m-1} \equiv G_{p+1}^{p^{m-1}} \pmod{p}$$

so that we can take $f_{m-1} := G_{p+1}^{p^{m-1}}$ that is of weight $k_{m-1} = p^{m-1}(p+1)$.

So, suppose that $j \leq m-2$. We claim that for $r = 0, \ldots, j$ there is a modular form g_r of weight $2 + p^{m-j-1}(p^{r+1} - 1)$, rational q-expansion with $v_p(g_r) = 0$, and such that

$$G_{2+p^{m-j-1}(p-1)} \mid V^r \equiv g_r \pmod{p^{m-j}}$$

which is the desired when $r = j$.

We prove the last claim by induction on r noting that the case $r = 0$ is trivial. So, suppose that $r < j$ and that we have already shown the existence of a modular form g_r as above. Notice that

$$p^{m-j+r} + 1 - (2 + p^{m-j-1}(p^{r+1} - 1)) = p^{m-j-1} - 1 \geq m - j$$

holds because $m - j \geq 2$ (we used here that $p > 2$). Thus we see that Lemma 2 applies (taking $s = m - j + r$) and shows the existence of a modular form g_{r+1} with rational q-expansion and $v_p(g_{r+1}) = 0$, such that

$$g_r \mid V \equiv g_{r+1} \pmod{p^{m-j}},$$

and such that g_{r+1} has weight

$$2 + p^{m-j-1}(p^{r+1} - 1) + p^{m-j+r}(p-1) = 2 + p^{m-j-1}(p^{r+2} - 1),$$

and we are done. □

Remark 1. *It is interesting to note that in the induction, the inequalities do not allow us to deal with the case $j = m-1$, but then we use the congruence $f \mid V \equiv f^p \pmod{p}$ to take care of the last term.*

Proof of Proposition 2: Again by Corollary 2 we have:

$$G_2 \equiv G_{2+p(p-1)} + p \cdot (G_{p+1} \mid V) \pmod{p^2}.$$

Noting again that $G_{p+1} \mid V \equiv G_{p+1}^p \pmod{p}$ so that

$$p \cdot (G_{p+1} \mid V) \equiv p \cdot G_{p+1}^p \pmod{p^2},$$

we are done. □

2.2. **Definition and properties of the θ operator.** Recall the classical θ operator acting on formal q-expansions as $q\frac{d}{dq}$, i.e.,

$$\theta\left(\sum a_n q^n\right) := \sum n a_n q^n.$$

If $f = \sum a_n q^n \in M_k(N, \mathbb{C})$ is a modular form of weight k then

$$\frac{1}{12}\partial f := \theta f - \frac{k}{12} E_2 \cdot f = \theta f + 2kG_2 \cdot f$$

is in $M_{k+2}(N, \mathbb{C})$ (we used $B_2 = \frac{1}{6}$ so that $E_2 = -24G_2$): This follows by writing $\theta = \frac{1}{2\pi i} \cdot \frac{d}{dz}$ (as $q = e^{2\pi i z}$) and by combining with the classical transformation properties of E_2 under the weight 2 action of $\mathrm{SL}_2(\mathbb{Z})$. Thus, we see that the operator ∂ defines a derivation on $M(N, \mathbb{Z}_p) := \oplus_k M_k(N, \mathbb{Z}_p)$.

If we now rewrite the above as $\theta f = \frac{1}{12}\partial f - 2kG_2 \cdot f$ and combine with Proposition 1, we obtain the following proposition.

Proposition 3. *Let $N, m \in \mathbb{N}$. For every $k \in \mathbb{N}$, there exists a linear map*

$$\theta : M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \longrightarrow M_{k+2}(N, \mathbb{Z}/p^m\mathbb{Z}) \oplus \bigoplus_{j=0}^{m-1} p^j M_{k+k_j}(N, \mathbb{Z}/p^m\mathbb{Z})$$

with the k_j as in Proposition 1, and whose effect on q-expansions is $\sum a_n q^n \mapsto \sum n a_n q^n$.

Clearly, applying Proposition 2 instead of Proposition 1 for the case $m = 2$, we have:

Proposition 4. *For every $k \in \mathbb{N}$ there exists a linear map*

$$\begin{aligned} \theta : \quad & M_k(N, \mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \\ & M_{k+2}(N, \mathbb{Z}/p^2\mathbb{Z}) \oplus M_{k+2+p(p-1)}(N, \mathbb{Z}/p^2\mathbb{Z}) \oplus pM_{k+p(p+1)}(N, \mathbb{Z}/p^2\mathbb{Z}) \end{aligned}$$

whose effect on q-expansions is $\sum a_n q^n \mapsto \sum n a_n q^n$.

One can see θ modulo p^m as mapping to the single weight $k + k(m)$ where

$$k(m) := 2 + 2p^{m-1}(p-1).$$

This is because, for each $j = 0, \ldots m-1$, we have – as one easily checks – that $k(m) - k_j$ is a multiple of $p^{m-j-1}(p-1)$ so that we have a natural inclusion

$$p^j M_{k+k_j}(N, \mathbb{Z}/p^m\mathbb{Z}) \hookrightarrow p^j M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z})$$

for each j.

Theorem 1. *The classical theta operator θ induces an operator*

$$\theta : M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z}),$$

where $k(m) = 2 + 2p^{m-1}(p-1)$.

The θ operator modulo p^m satisfies the same simple commutation rules with Hecke operators T_ℓ ($\ell \neq p$ prime) that are well-known in the case $m = 1$.

Proposition 5. *Let $k, m \in \mathbb{N}$ and let ℓ be a prime with $\ell \neq p$. Then*

$$T_\ell \theta = \ell \cdot \theta T_\ell$$

as linear maps $M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to \bigoplus_{j=0}^{m-1} p^j M_{k+k_j}(N, \mathbb{Z}/p^m\mathbb{Z}) \to \mathbb{Z}/p^m\mathbb{Z}[[q]]$, where the last arrow denotes the q-expansion map. Consequently, we have $T_\ell \theta = \ell \theta T_\ell$ as linear maps $M_k(N, \mathbb{Z}/p^m\mathbb{Z}) \to M_{k+k(m)}(N, \mathbb{Z}/p^m\mathbb{Z})$.

Proof. Let $f \in M_k(N, \mathbb{Z}/p^m\mathbb{Z})$ with q-expansion $\sum_n a_n q^n$. If we denote by $\sum_n a_n^{(j)} q^n$ the q-expansion of the j'th component of θf then it follows that we have

$$\sum_{j=0}^{m-1} a_n^{(j)} = n a_n$$

for all n. Furthermore, the q-expansion of $T_\ell \theta f$ is

$$\sum_n \left(\sum_{j=0}^{m-1} a_{\ell n}^{(j)} + \ell^{k+k_j-1} a_{\frac{n}{\ell}}^{(j)} \right) q^n$$

(with the usual convention that $a_{\frac{n}{\ell}}^{(j)} := 0$ if $\ell \nmid n$.)

Now, since $v_p(a_n^{(j)}) \geq j$ for all n and each j, since $k_j \equiv 2 \pmod{p^{m-j-1}(p-1)}$, and since $\ell \neq p$, we find

$$\ell^{k+k_j-1} a_{\frac{n}{\ell}}^{(j)} \equiv \ell^{k+1} a_{\frac{n}{\ell}}^{(j)} \pmod{p^m}.$$

Hence, the n'th coefficient of the q-expansion of $T_\ell \theta f$ equals

$$\sum_{j=0}^{m-1} \left(a_{\ell n}^{(j)} + \ell^{k+1} a_{\frac{n}{\ell}}^{(j)} \right) = \ell n a_{\ell n} + \ell^{k+1} \cdot \frac{n}{\ell} a_{\frac{n}{\ell}} = \ell n \left(a_{\ell n} + \ell^{k-1} a_{\frac{n}{\ell}} \right)$$

which is precisely ℓ times the n'the coefficient of the q-expansion of $\theta T_\ell f$. □

3. Weight filtrations on packages of modular forms

For even $k \in \mathbb{N}$, the monomials $Q^a R^b$ such that $4a + 6b = k$ form a $\mathbb{Z}_p$-basis for $M_k(\mathbb{Z}_p)$ and we have that $M(\mathbb{Z}_p) := \oplus_k M_k(\mathbb{Z}_p) \cong \mathbb{Z}_p[Q, R]$ as a $\mathbb{Z}_p$-algebra, where $Q = E_4$ and $R = E_6$ are the normalized Eisenstein series of weight 4 and 6, respectively. Furthermore, $\mathbb{Z}_p[Q, R] \cong \mathbb{Z}_p[X, Y]$, where $\mathbb{Z}_p[X, Y]$ is the polynomial ring over $\mathbb{Z}_p$ in the variables X and Y. Thus,

$$\begin{aligned} M(\mathbb{Z}_p) = \oplus_k M_k(\mathbb{Z}_p) &\cong \mathbb{Z}_p[X, Y] \\ M_k(\mathbb{Z}_p) &\cong \sum_{4a+6b=k} \mathbb{Z}_p \cdot X^a Y^b, \end{aligned}$$

and

$$M(\mathbb{Z}/p^m\mathbb{Z}) = \oplus_k M_k(\mathbb{Z}/p^m\mathbb{Z}) \cong \mathbb{Z}/p^m\mathbb{Z}[X,Y]$$
$$M_k(\mathbb{Z}/p^m\mathbb{Z}) \cong \sum_{4a+6b=k} \mathbb{Z}/p^m\mathbb{Z}\cdot X^aY^b,$$

where the $k \in \mathbb{Z}_{\geq 0}$ are even. Hence, $M(\mathbb{Z}_p) \cong \mathbb{Z}_p[X,Y]$ (resp. $M(\mathbb{Z}/p^m\mathbb{Z}) \cong (\mathbb{Z}/p^m\mathbb{Z})[X,Y]$) is a graded $\mathbb{Z}_p$-algebra (resp. $\mathbb{Z}/p^m\mathbb{Z}$-algebra), where the grading is given by the weight. An element in $M(\mathbb{Z}_p)$ (resp. $M(\mathbb{Z}/p^m\mathbb{Z})$) is said to be isobaric of weight k if it is a $\mathbb{Z}_p$-linear (resp. $\mathbb{Z}/p^m\mathbb{Z}$-linear) combination of monomials of weight k where X and Y are defined to be of weight 4 and 6, respectively.

Let $1 \leq t \leq m$. Then p^tF is isobaric of weight k in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ if and only if F is isobaric of weight k in $(\mathbb{Z}/p^{m-t}\mathbb{Z})[X,Y]$,

Theorem 2. *If $f_1, f_2 \in M_k(\mathbb{Z}/p^m\mathbb{Z})$ or $M_k(\mathbb{Z}_p)$ have the same q-expansion in $(\mathbb{Z}/p^m\mathbb{Z})[[q]]$ or $\mathbb{Z}_p[[q]]$, respectively, then they are equal.*

Proof. This is the q-expansion principle. □

Although we shall not use the following theorem directly, we provide it here for completeness.

Theorem 3. *Assume $p \geq 5$. Let $M(\mathbb{Z}/p^2\mathbb{Z}) = \oplus_k M_k(\mathbb{Z}/p^2\mathbb{Z})$ be the ring of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ over $\mathbb{Z}/p^2\mathbb{Z}$. The image of $M(\mathbb{Z}/p^2\mathbb{Z})$ under the q-expansion map, which lies in $(\mathbb{Z}/p^2\mathbb{Z})[[q]]$, is isomorphic to $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]/\mathfrak{a}_2$, where $\mathfrak{a}_2 = (p(A-1),(A-1)^2)$, $E_{p-1} = A(Q,R)$, and $A \in \mathbb{Z}_p[X,Y]$.*

Proof. Katz' description of the ideal of congruences of higher order [5, Theorem 4.4] shows that the kernel of the map $M(\mathbb{Z}_p) \to \mathbb{Z}_p[[q]] \to \mathbb{Z}/p^2\mathbb{Z}[[q]]$ is given by

$$I_2 = ((E_{p-1}-1)^2, p(E_{p-1}-1), p^2).$$

Since $M(\mathbb{Z}_p) \cong \mathbb{Z}_p[X,Y]$ and $M(\mathbb{Z}/p^2\mathbb{Z}) \cong M(\mathbb{Z}_p) \otimes \mathbb{Z}/p^2\mathbb{Z} \cong (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, the result follows. □

Lemma 3. *Let $A \in (\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ be nonzero in $\mathbb{F}_p[X,Y]$.*
If $AH = 0$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$, then $H = 0$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$.

Proof. The assertion is true for $m = 1$ as $\mathbb{F}_p[X,Y]$ is an integral domain. Suppose $AH = 0$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$. Then $AH = 0$ in $\mathbb{F}_p[X,Y]$ so that $H = 0$ in $\mathbb{F}_p[X,Y]$. Thus, $H = pH_1$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$, and since $AH = ApH_1 = 0$ in $\mathbb{Z}/p^m\mathbb{Z}$, we have that $AH_1 = 0$ in $(\mathbb{Z}/p^{m-1}\mathbb{Z})[X,Y]$. By the induction, $H_1 = 0$ in $(\mathbb{Z}/p^{m-1}\mathbb{Z})[X,Y]$, so $H = pH_1 = 0$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$. □

Lemma 4. *Let $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ be graded by the weight, i.e. X has weight 4 and Y has weight 6. Suppose $A \in (\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ is isobaric of weight k_0 and A is nonzero in $\mathbb{F}_p[X,Y]$. Suppose F is isobaric of weight k and $F = AG$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$. Then G is isobaric of weight $k - k_0$.*

Proof. Write $G = G_1 + \ldots + G_t$ where $G_i \in (\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ is nonzero and isobaric of weight k_i, where $k_1 < \ldots < k_t$. Then

$$\begin{aligned} F &= AG \\ &= A(G_1 + \ldots + G_t) \\ &= AG_1 + \ldots + AG_t. \end{aligned}$$

By Lemma 3, none of the $AG_i = 0$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$. Each AG_i is isobaric of weight $k_i + k_0$. Comparing both sides, we obtain that $t = 1$ and $F = AG$, where $G = G_1$ is isobaric of weight $k_1 = k - k_0$. □

Theorem 4. *Suppose for $i = 1, 2$, we have that $f_i \in M_{k_i}(\mathbb{Z}_p)$, $f_i \not\equiv 0 \pmod p$, $k_2 > k_1$, and $f_2 \equiv f_1 \pmod{p^m}$. Write $f_i = F_i(Q,R)$ and $E_{p-1} = A(Q,R)$, where $F_i, A \in \mathbb{Z}_p[X,Y]$. Then $k_2 = k_1 + p^{m-1}(p-1)t$ and*

$$F_2 \equiv F_1 \left(A^{p^{m-1}}\right)^t \pmod{p^m}$$

for some $t > 0$.

Proof. It is known that the congruence $f_2 \equiv f_1 \pmod{p^m}$, together with the conditions $f_i \not\equiv 0 \pmod p$ and $k_2 > k_1$, implies that $k_2 = k_1 + p^{m-1}(p-1)t$ for some $t > 0$, assuming $k_2 > k_1$ (cf. [8, Remarque après Théorème 2] or [2]). Now,

$$f_2 \equiv f_1 \equiv f_1 \left(E_{p-1}^{p^{m-1}}\right)^t \pmod{p^m}.$$

Since f_2 and $f_1 \left(E_{p-1}^{p^{m-1}}\right)^t$ both have weight k_2, by the q-expansion principle they are the same in $M_k(\mathbb{Z}/p^m\mathbb{Z})$, and hence

$$F_2 \equiv F_1 \left(A^{p^{m-1}}\right)^t \pmod{p^m}.$$

□

If $f \in M_k(\mathbb{Z}/p^m\mathbb{Z})$, $f \not\equiv 0 \pmod p$, we define the weight of f modulo p^m as

$$w_{p^m}(f) = \min\left\{k_0 \in \mathbb{Z}_{\geq 0} : f \equiv h \pmod{p^m}, h \in M_{k_0}(\mathbb{Z}/p^m\mathbb{Z})\right\}. \tag{1}$$

For $f \in M_k(\mathbb{Z}/p^m\mathbb{Z})$, $f \not\equiv 0 \pmod p$, we have that $w_{p^m}(f) = k$ if and only if

$$f \notin M_{k-p^{m-1}(p-1)}(\mathbb{Z}/p^m\mathbb{Z}) \hookrightarrow M_k(\mathbb{Z}/p^m\mathbb{Z}).$$

Theorem 4 allows us to deduce the following facts concerning the weight of $f \in M_k(\mathbb{Z}/p^m\mathbb{Z})$.

Corollary 3. *Let $f \in M_k(\mathbb{Z}/p^m\mathbb{Z})$, $f \not\equiv 0 \pmod p$. Then $w_{p^m}(f) \equiv k \pmod{p^{m-1}(p-1)}$.*

Corollary 4. *Let $f \in M_k(\mathbb{Z}/p^m\mathbb{Z})$, $f \not\equiv 0 \pmod p$, and write $f = F(Q,R)$ and $E_{p-1} = A(Q,R)$, where $f, A \in (\mathbb{Z}/p^m\mathbb{Z})[X,Y]$. Then $w_{p^m}(f) = k - tp^{m-1}(p-1)$ if and only if $A^{p^{m-1}t} \mid F$, $A^{p^{m-1}(t+1)} \nmid F$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$, where $t \geq 0$.*

Proof. If $w_{p^m}(f) = k - tp^{m-1}(p-1)$, then $f \equiv g \pmod{p^m}$ for some

$$g \in M_{k-tp^{m-1}(p-1)}(\mathbb{Z}/p^m\mathbb{Z}),$$

with $f, g \not\equiv 0 \pmod p$. By Theorem 4, $A^{p^{m-1}t} \mid F$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$.

Conversely, suppose $A^{p^{m-1}t} \mid F$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$. Note that A is non-zero in $\mathbb{F}_p[X,Y]$. Then by Lemma 4, $F = A^{p^{m-1}t}G$ in $(\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ for some $G \in (\mathbb{Z}/p^m\mathbb{Z})[X,Y]$ which is isobaric of weight $k - tp^{m-1}(p-1)$. Thus, $f \equiv g \pmod{p^m}$ for some $g \in M_{k-tp^{m-1}(p-1)}(\mathbb{Z}/p^m\mathbb{Z})$. □

We will need the following facts concerning divisibility in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$.

Proposition 6. *Let $A, F, F_0, H_0 \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, where A is nonzero in $\mathbb{F}_p[X,Y]$, $s \geq t \geq 0$, $s_1 \geq 0$. Then we have the following:*

(1) *$A^s \mid pF$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ if and only if $A^s \mid F$ in $\mathbb{F}_p[X,Y]$.*
(2) *$A^s \mid F$, $A^{s+1} \nmid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, and $A^{s+s_1} \mid F$, $A^{s+s_1+1} \nmid F$ in $\mathbb{F}_p[X,Y]$ if and only if $F = A^{s+s_1}F_0 + pA^sH_0$, where $A \nmid F_0, H_0$ in $\mathbb{F}_p[X,Y]$.*
(3) *$A^s \mid A^tF$ in $\mathbb{F}_p[X,Y]$ if and only if $A^{s-t} \mid F$ in $\mathbb{F}_p[X,Y]$.*
(4) *$A^s \mid A^tF$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ if and only if $A^{s-t} \mid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$.*

Proof. For the first assertion, note that $A^s \mid pF$ if and only if $pF = A^sH$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. But A is nonzero in $\mathbb{F}_p[X,Y]$, so by Lemma 3 we must have that $H = pK$ for some $K \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Thus, $A^s \mid pF$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ if and only if $pF = A^sH = A^spK$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ if and only if $F = A^sK$ in $\mathbb{F}_p[X,Y]$

For the second assertion, write $F = A^{s+s_1}F_0 + pH$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. If $A^s \mid F$ in $\mathbb{Z}/p^2\mathbb{Z}$, then $A^s \mid pH$, so $A^s \mid H$ in $\mathbb{F}_p[X,Y]$ and hence $F = A^{s+s_1}F_0 + pA^sH_0$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. If $A \mid H_0$ in $\mathbb{F}_p[X,Y]$, then $A^{s+1} \mid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ (for $s_1 \geq 1$, this follows immediately, otherwise $F = A^sF_0 + pA^sH_0 = A^s(F_0 + pF_0) + pA^s(H_0 - F_0)$ is of the required form). Conversely, if $F = A^{s+s_1}F_0 + pA^sH_0$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, where $A \nmid F_0, H_0$ in $\mathbb{F}_p[X,Y]$, then $A^s \mid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, and $A^{s+1} \nmid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ (since if $A^{s+1} \mid F$, then $A \mid (A^{s_1}F_0 + pH_0)$, so either $s_1 \geq 1$ and $A \mid H_0$ in $\mathbb{F}_p[X,Y]$ or $s_1 = 0$ and $A \mid F_0$ in $\mathbb{F}_p[X,Y]$).

The third assertion follows from the fact that $\mathbb{F}_p[X,Y]$ is an integral domain. Now if $A^s \mid A^tF$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, so that $A^tF = A^sH$ for some $H \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, and hence $F = A^{s-t}F_0 + pH$ for some $F_0, H \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ by applying Lemma 3. Thus, $A^tF = A^sF_0 + pA^tH$. Now, $A^s \mid A^tF$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ implies that $A^s \mid pA^tH$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, which in turn implies that $A^s \mid A^tH$ in $\mathbb{F}_p[X,Y]$, whence $A^{s-t} \mid H$ in $\mathbb{F}_p[X,Y]$ and the finally $A^{s-t} \mid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. ☐

Proposition 7. *Let $f \in M_k(\mathbb{Z}/p^2\mathbb{Z})$, $f \not\equiv 0 \pmod{p}$, and write $f = F(Q,R)$ and $E_{p-1} = A(Q,R)$, where $F, A \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Then $F = A^{s+s_1}F_0 + pA^sH_0$ for some $s, s_1 \geq 0$, $F_0, H_0 \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, $A \nmid F_0, H_0$ in $\mathbb{F}_p[X,Y]$. Furthermore, F_0 can be taken to be isobaric of weight $k-(s+s_1)(p-1)$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ and H_0 isobaric of weight $k-s(p-1)$ in $\mathbb{F}_p[X,Y]$.*

Proof. Note first that A is nonzero in $\mathbb{F}_p[X,Y]$. Since $f \not\equiv 0 \pmod{p}$, we have that F is non-zero in $\mathbb{F}_p[X,Y]$. Thus, there exist $s, s_1 \geq 0$ such that $A^s \mid F$, $A^{s+1} \nmid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, and $A^{s+s_1} \mid F$, $A^{s+s_1+1} \nmid F$ in $\mathbb{F}_p[X,Y]$. By Proposition 6, we can write $F = A^{s+s_1}F_0 + pA^sH_0$, where $A \nmid F_0, H_0$ in $\mathbb{F}_p[X,Y]$.

We claim that we can modify F_0, H_0 in the above expression for F so that F_0 is isobaric of weight $k-(s+s_1)(p-1)$ and H_0 is isobaric of weight $k-s(p-1)$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Since F, A are isobaric of weights k and $p-1$, respectively, and $F = A^{s+s_1}F_0$ in $\mathbb{F}_p[X,Y]$, we conclude that F_0 is isobaric of weight $k-(s+s_1)(p-1)$ in $\mathbb{F}_p[X,Y]$ by Lemma 4. Thus, $F_0 = F_1 + pF_2$ where F_1 is isobaric of weight $k-(s+s_1)(p-1)$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Hence, $F = A^{s+s_1}F_1 + pA^sH_1$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ as before, where $H_1 = H_0 + A^{s_1}F_2$, but now F_1 is isobaric of weight $k-(s+s_1)(p-1)$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Finally, H_1 is isobaric of weight $k-s(p-1)$ in $\mathbb{F}_p[X,Y]$ as $F - A^{s+s_1}F_1 = pA^sH_1$, so that A^sH_1 is isobaric of weight k in $\mathbb{F}_p[X,Y]$. By Lemma 4, H_1 is isobaric of weight $k-s(p-1)$ in $\mathbb{F}_p[X,Y]$. ☐

For $f \in M_k(\mathbb{Z}/p^2\mathbb{Z})$, $f \not\equiv 0 \pmod{p}$, write $f = F(Q,R)$ and $E_{p-1} = A(Q,R)$, where $F, A \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. If $F = A^{s+s_1}F_0 + pA^sH_0$, $A \nmid F_0, H_0$ in $\mathbb{F}_p[X,Y]$, then we define the mass of f to be $m_{p^2}(f) := (s+s_1, s)$. This is well-defined by Proposition 7 and 6.

Although we do not refer to the below lemma directly, the principle is used repeatedly in later proofs.

Lemma 5. *Let $f \in M_{k_1}(\mathbb{Z}/p^2\mathbb{Z}), g \in K_{k_2}(\mathbb{Z}/p^2\mathbb{Z}), f, g \not\equiv 0 \pmod{p}$. Suppose $m_{p^2}(f) = (s+s_1, s), m_{p^2}(g) = (t+t_1, t)$. Let us lexicographically order masses. Then $m_{p^2}(fg) \geq (s+s_1+t+t_1, s+t+\min\{s_1, t_1\})$; equality holds if $s_1 \neq t_1$.*

Proof. Let $F = A^{s+s_1}F_0 + pA^sF_1$, $G = A^{t+t_1}G_0 + pA^tG_1$, where $A \nmid F_0, G_0, F_1, G_1$ in $\mathbb{F}_p[X, Y]$. Then

$$\begin{aligned} FG &= (A^{s+s_1}F_0 + pA^sF_1)(A^{t+t_1}G_0 + pA^tG_1) \\ &= A^{s+s_1+t+t_1}F_0G_0 + pA^{s+t+t_1}F_1G_0 + pA^{s+t+s_1}F_0G_1 \\ &= A^{s+s_1+t+t_1}F_0G_0 + pA^{s+t}(A^{t_1}F_1G_0 + pA^{s_1}F_0G_1). \end{aligned}$$

□

Corollary 5. *Let $f \in M_k(\mathbb{Z}/p^2\mathbb{Z})$, and write $f = F(Q, R)$ and $E_{p-1} = A(Q, R)$, where $F, A \in (\mathbb{Z}/p^2\mathbb{Z})[X, Y]$. Suppose that $F = A^{s+s_1}F_0 + pA^sH_0$ for some $s, s_1 \geq 0$, $F_0, H_0 \in (\mathbb{Z}/p^2\mathbb{Z})[X, Y]$, $A \nmid F_0, H_0$ in $\mathbb{F}_p[X, Y]$. Then we have that*

$$\begin{aligned} w_{p^2}(f) &= k - \lfloor s/p \rfloor p(p-1) \\ w_p(f) &= k - (s+s_1)(p-1). \end{aligned}$$

Proof. If we put $t := \lfloor s/p \rfloor$, then $A^{pt} \mid F$ but $A^{p(t+1)} \nmid F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X, Y]$ by Proposition 6, and hence $w_{p^2}(f) = k - \lfloor s/p \rfloor p(p-1)$ by Corollary 4. Similarly, $w_p(f) = k - (s+s_1)(p-1)$. □

Proposition 8. *Suppose $A, F \in (\mathbb{Z}/p^2\mathbb{Z})[X, Y]$, where A is nonzero in $\mathbb{F}_p[X, Y]$, and $F = A^{s+s_1}F_0 + pA^sH_0$, $F_0, H_0 \in (\mathbb{Z}/p^2\mathbb{Z})[X, Y]$, $A \nmid F_0, H_0$ in $\mathbb{F}_p[X, Y]$, $s, s_1 \geq 0$. Assume also that A is coprime to ∂A in $\mathbb{F}_p[X, Y]$.*

Then, if $1 \leq s \leq p-1$, we have that $A^{s-1} \mid \partial F$, but $A^s \nmid \partial F$, in $(\mathbb{Z}/p^2\mathbb{Z})[X, Y]$. In particular, $A^p \nmid \partial F$.

Proof. We find

$$\partial F = A^{s+s_1-1}\left((s+s_1)(\partial A)F_0 + A\partial F_0\right) + pA^{s-1}\left(s(\partial A)H_0 + A\partial H_0\right).$$

Suppose first that $p \nmid s+s_1$. Then A divides neither $(s+s_1)(\partial A)F_0 + A\partial F_0$ nor $s\partial H_0 + A\partial H$ in $\mathbb{F}_p[X, Y]$ (as $p \nmid s$). Proposition 6 (2) then shows that $A^{s-1} \mid \partial F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X, Y]$, but $A^s \nmid \partial F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X, Y]$.

If $p \mid s+s_1$ we must have $s_1 \geq 1$ as $s \leq p-1$. Writing then

$$\partial F = A^{s+s_1}\partial F_0 + pA^{s-1}\left(\frac{s+s_1}{p}A^{s_1}(\partial A)F_0 + s(\partial A)H_0 + A\partial H_0\right)$$

we have that the contents of the parenthesis is not divisible by A in $\mathbb{F}_p[X, Y]$. Thus, if ∂F_0 is nonzero in $\mathbb{F}_p[X, Y]$, the desired follows from Proposition 6 (2) whereas it follows from Proposition 6 (1) if $\partial F_0 = 0$. □

Theorem 5. *Let $f \in M_k(\mathbb{Z}/p^2\mathbb{Z})$, $f \not\equiv 0 \pmod{p}$, $f = F(Q, R)$, $E_{p-1} = A(Q, R)$, $E_{p(p-1)+2} = B(Q, R)$, $E_{p+1} = C(Q, R)$ where $F, A, B, C \in \mathbb{Z}_p[X, Y]$.*

Write $F = A^{s+s_1}F_0 + pA^sH_0$ in $(\mathbb{Z}/p^2\mathbb{Z})[X, Y]$, where $s, s_1 \geq 0$, $A \nmid F_0, H_0$ in $\mathbb{F}_p[X, Y]$.

Assume that $w_{p^2}(f) = k$ and $\theta f \not\equiv 0 \pmod{p}$. Then $0 \leq s \leq p-1$.

(i) Suppose that $k \not\equiv 0 \pmod p$. Then $w_{p^2}(\theta f) \le k + 2p(p-1) + 2$ with equality if and only if $w_p(f) = k$ or $k-(p-1)$ which again is equivalent to $A^2 \nmid F$ in $\mathbb{F}_p[X,Y]$.

If $A^2 \mid F$ in $\mathbb{F}_p[X,Y]$ and $s_1 \neq p+1$, write $w_p(f) = k - t(p-1)$, where $t = s + s_1 \ge 2$.

Then

$$w_{p^2}(\theta f) \begin{cases} = k + p(p-1) + 2 & \text{if } t < p + \max\{s_1 - (p-1), 2\} \\ \le k + 2 & \text{if } t \ge p + \max\{s_1 - (p-1), 2\}. \end{cases}$$

If $A^2 \mid F$ in $\mathbb{F}_p[X,Y]$ and $s_1 = p+1$, let b and c be the constant terms of $G_{p(p-1)+2}$ and G_{p+1}, respectively (which are nonzero in $\mathbb{F}_p$), and let $s_3 \ge 0$ is the highest power of A dividing $bH_0 + c^pC^{p-1}F_0$ in $\mathbb{F}_p[X,Y]$. Then

$$w_{p^2}(\theta f) \begin{cases} = k + p(p-1) + 2 & \text{if } s = s_3 = 0 \\ \le k + 2 & \text{otherwise.} \end{cases}$$

(ii) Suppose that $k \equiv 0 \pmod p$, but $k \not\equiv 0 \pmod{p^2}$. Then $w_{p^2}(\theta f) \le k + p(p-1) + 2$, and $w_{p^2}(\theta f) = k + p(p-1) + 2$ if and only if $w_p(f) = k$ which again is equivalent to $s = s_1 = 0$.

If $s \ge 1$, then $w_{p^2}(f) = k + 2$.

(iii) Suppose that $k \equiv 0 \pmod{p^2}$. Then $w_{p^2}(\theta f) \le k + 2$. If in addition we have $s \ge 1$, then $w_{p^2}(f) = k + 2$.

Proof. Note that A is nonzero in $\mathbb{F}_p[X,Y]$ as $E_{p-1} \neq 0$ in $M_{p-1}(\mathbb{Z}/p\mathbb{Z})$. The condition $w_{p^2}(f) = k$ implies $0 \le s \le p-1$ by Corollary 4. Obviously, we have $w_{p^2}(\theta f) \le k + 2p(p-1) + 2$ in any case. Also, the condition $\theta f \not\equiv 0 \pmod p$ means that we $w_{p^2}(f) \equiv k \pmod{p(p-1)}$.

Set $P = E_2$. We know that $\theta f = \frac{k}{12} P f + \frac{1}{12} \partial f$ where ∂ is the derivation of $M(\mathbb{Z}_p) \cong \mathbb{Z}_p[X,Y]$ such that $\partial Q = -4R$ and $\partial R = -6Q^2$. Since $E_{p-1} \equiv 1 \pmod p$ we have that $\theta E_{p-1} \equiv 0 \pmod p$ so that $(p-1)PE_{p-1} + \partial E_{p-1} \equiv 0 \pmod p$ and hence $\partial E_{p-1} \equiv PE_{p-1} \equiv P \equiv E_{p+1} \equiv E_{p(p-1)+2} \pmod p$. Since $E_{p-1}^{p-1} \partial E_{p-1} \equiv E_{p(p-1)+2} \pmod p$ both have weight $p(p-1)+2$, we have that $A^{p-1}\partial A = B$ in $\mathbb{F}_p[X,Y]$ by the q-expansion principle. Similarly, $E_{p+1} \equiv \partial E_{p-1} \pmod p$ both have weight $p+1$, so $\partial A = C$ in $\mathbb{F}_p[X,Y]$ by the q-expansion principle.

Now, $\partial^2 A(Q,R) \equiv \partial(PE_{p-1}) \equiv \partial P \cdot E_{p-1} + P \partial E_{p-1} \pmod p$. Since $\frac{2}{12}P^2 + \frac{1}{12}\partial P = \theta P = \frac{1}{12}(P^2 - Q)$, we have that $\partial P = -P^2 - Q$ so that $\partial^2 A = -QA$ in $\mathbb{F}_p[X,Y]$, since $P^2 E_{p-1} \equiv E_{p+1}P \pmod p$ have the same weight. Thus, A satisfies a second order partial differential equation in $\mathbb{F}_p[X,Y]$ and according to Serre [8, Corollaire 1] this implies by a standard argument of Igusa that A is coprime to $\partial A = C$ in $\mathbb{F}_p[X,Y]$. (Alternatively, see [7] Theorem 7.3, p. 167). Hence, $B = A^{p-1}\partial A$ in $\mathbb{F}_p[X,Y]$ and is not divisible by A^p in $\mathbb{F}_p[X,Y]$.

Let $g = \frac{1}{12}\partial f = \theta f - \frac{k}{12} P f \in M_{k+2}(\mathbb{Z}/p^2\mathbb{Z})$. Write

$$\begin{aligned} f &= F(Q,R), \\ g &= G(Q,R) = \frac{1}{12}\partial F(Q,R) = \frac{1}{12}\partial f, \\ \theta f &= H(Q,R), \end{aligned}$$

where $F, G, H \in (\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Then

$$H = \Big(-2kbBF + A^p \frac{1}{12}\partial F \Big) A^p - 2kpc^pC^pFA^{p-2},$$

(recall that b and c are the constant terms of $G_{p(p-1)+2}$ and G_{p+1}, respectively).

(i) Suppose $k \not\equiv 0 \pmod p$. By Corollary 4 we have $w_{p^2}(\theta f) = k + 2p(p-1) + 2$ if and only if $A^p \nmid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$.

Now, if $A^2 \nmid F$ in $\mathbb{F}_p[X,Y]$, then H is not divisible by A^p in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ by Proposition 6. Hence, $w_{p^2}(\theta f) = k + 2p(p-1) + 2$. On the other hand, if $A^2 \mid F$ in $\mathbb{F}_p[X,Y]$, then $A^p \mid 2kpc^pC^pFA^{p-2}$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, and hence $A^p \mid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$.

In other words, $w_{p^2}(\theta f) = k + 2p(p-1) + 2$ if and only if $A^2 \nmid F$ in $\mathbb{F}_p[X,Y]$. By Corollary 4 this condition is equivalent to $w_p(f) = k$ or $w_p(f) = k - (p-1)$.

Now, suppose $A^2 \mid F$ in $\mathbb{F}_p[X,Y]$, which corresponds to $w_p(f) = k - t(p-1)$, where $t = s + s_1 \geq 2$. Then

$$\begin{aligned} H &= A^p\left(-2kbB(A^{s+s_1}F_0 + pA^sH_0) + A^p\frac{1}{12}\partial F - 2kpc^pC^pA^{s+s_1-2}F_0\right) \\ &= A^p\left(-2kbBA^{s+s_1}F_0 - 2kpbA^{s+p-1}CH_0 - 2kpc^pC^pA^{s+s_1-2}F_0 + A^p\frac{1}{12}\partial F\right). \end{aligned}$$

Let

$$\begin{aligned} K &= -2kbBA^{s+s_1}F_0 - 2kpbA^{s+p-1}CH_0 - 2kpc^pC^pA^{s+s_1-2}F_0 \\ &= -2kbBA^{s+s_1}F_0 - pL_0 \\ L_0 &= 2kbA^{s+p-1}CH_0 + 2kc^pC^pA^{s+s_1-2}F_0. \end{aligned}$$

Suppose $s_1 \neq p+1$. Then $s+p-1 \neq s+s_1-2$. Put $s_2 := s-2+\min\{p+1, s_1\}$ so that s_2 is either $s+s_1-2$ or $s+p-1$. Note that $0 \leq s_2 := s-2+\min\{p+1,s_1\} \leq s+s_1-2 < s+s_1$. We deduce that $A^{s_2} \mid L_0$, $A^{s_2+1} \nmid L_0$ in $\mathbb{F}_p[X,Y]$ so $A^{s_2} \mid K$, $A^{s_2+1} \nmid K$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ by Proposition 6.

Writing $s_2 = s + s_1 + \min\{p-1-s_1, -2\} = t + \min\{p-1-s_1,-2\}$ we have now $w_{p^2}(\theta f) = k + p(p-1) + 2$ if $t + \min\{p-1-s_1,-2\} < p$, and $w_{p^2}(\theta f) \leq k+2$ if $t + \min\{p-1-s_1,-2\} \geq p$, again using Corollary 4. The result follows now from the relation $-\min\{a,b\} = \max\{-a,-b\}$.

If $s_1 = p+1$ so that $s+p-1 = s+s_1-2$, then by definition of $s_3 \geq 0$ the highest power of A which divides $L_0 = 2kCA^{s+p-1}\left(bH_0 + c^pC^{p-1}F_0\right)$ in $\mathbb{F}_p[X,Y]$ is $s+p-1+s_3$. Thus, if $s+p-1+s_3 < p$, then $w_{p^2}(\theta f) = k+2+p(p-1)$, and otherwise $w_{p^2}(\theta f) \leq k+2$.

(ii) Suppose $k \equiv 0 \pmod p$, but $k \not\equiv 0 \pmod{p^2}$. Then

$$\begin{aligned} H &= -2kbBFA^p + A^{2p}\frac{1}{12}\partial F \\ &= -2kbA^{2p-1+s+s_1}CF_0 + A^{2p}\frac{1}{12}\partial F. \end{aligned}$$

using $B = A^{p-1}\partial A$ and $\partial A = C$ in $\mathbb{F}_p[X,Y]$. We see that $w_{p^2}(\theta f) \leq k+p(p-1)+2$ with equality if and only if $A^{2p} \nmid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, by Corollary 4.

Now, $A^{2p} \mid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ if and only if $s + s_1 > 0$ if and only if $A \mid F$ in $\mathbb{F}_p[X,Y]$ by Proposition 6. Hence, $w_{p^2}(\theta f) = k + p(p-1) + 2$ if and only if $w_p(f) = k$.

Suppose now that $s \geq 1$ so that $w_{p^2}(\theta f) \leq k+2$ with equality if and only if $A^{3p} \nmid H$.

By Proposition 8 we now have $A^{s-1} \mid \partial F$ but $A^s \nmid \partial F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$. Thus, if $s_1 = 0$ we see that $A^{2p-1+s} \mid H$, $A^{2p+s} \nmid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ by Proposition 6. As $s \le p-1$ we thus have $A^{3p} \nmid H$.

On the other hand, if $s_1 \ge 1$ then $A^{2p+s} \mid 2kbA^{2p-1+s+s_1}CF_0$ but $A^{2p+s} \nmid A^{2p}\frac{1}{12}\partial F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, and we have again that $A^{2p+s} \nmid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$.

(iii) Suppose then that $k \equiv 0 \pmod{p^2}$. Then $\theta f = \frac{1}{12}\partial f$, $H = A^{2p}\frac{1}{12}\partial F$, and $w_{p^2}(\theta f) \le k+2$. If $s \ge 1$, then by Proposition 8, $A^p \nmid \partial F$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$, so that $A^{3p} \nmid H$ in $(\mathbb{Z}/p^2\mathbb{Z})[X,Y]$ by Proposition 6, and therefore $w_{p^2}(F) = k+2$. □

Corollary 6. *Suppose $p \nmid k$. Then the kernel of*

$$\theta : M_k(\mathbb{Z}/p^2\mathbb{Z}) \to M_{k+2+2p(p-1)}(\mathbb{Z}/p^2\mathbb{Z})/M_{k+2+p(p-1)}(\mathbb{Z}/p^2\mathbb{Z})$$

is given by the submodule $A^2M_{k-2(p-1)}(\mathbb{Z}/p^2\mathbb{Z}) + pM_k(\mathbb{Z}/p^2\mathbb{Z})$. In particular, $p\theta(f) \in M_{k+2+p(p-1)}(\mathbb{Z}/p^2\mathbb{Z})$ for any $f \in M_k(\mathbb{Z}/p^2\mathbb{Z})$.

Proof. The assumption $p \nmid k$ means that we cannot have $\theta f \equiv 0 \pmod{p}$ using properties of θ modulo p.

If $f \equiv 0 \pmod{p}$, then $F = pF_1$ so that

$$\begin{aligned} H &= \Big(-2kbBF + A^p\frac{1}{12}\partial F\Big)A^p - 2kpc^pC^pFA^{p-2}, \\ &= \Big(-2kbBpF_1 + A^p\frac{1}{12}\partial pF_1\Big)A^p \\ &= \Big(-2kbA^{p-1}\partial ApF_1 + A^p\frac{1}{12}\partial pF_1\Big)A^p \\ &= pA^{2p-1}\Big(-2kb\partial AF_1 + A\frac{1}{12}\partial F_1\Big). \end{aligned}$$

Thus, $A^p \mid H$.

If $f \not\equiv 0 \pmod{p}$, then we apply Theorem 5 (i), which shows that $w_{p^2}(\theta f) \le k+2+p(p-1)$ if and only if $A^2 \mid F$ in $\mathbb{F}_p[X,Y]$. □

This suggests the module $M_k(\mathbb{Z}/p^2\mathbb{Z})/M_{k-p(p-1)}(\mathbb{Z}/p^2\mathbb{Z})$ is not free over $\mathbb{Z}/p^2\mathbb{Z}$. In fact, this can be seen directly by considering the element

$$f = E_{p-1}^p f_0 + ph_0 \in M_k(\mathbb{Z}/p^2\mathbb{Z}),$$

where $f_0 \in M_{k-p(p-1)}(\mathbb{Z}/p^2\mathbb{Z})$, $f_0 \not\equiv 0 \pmod{p}$, $h_0 \in M_k(\mathbb{Z}/p^2\mathbb{Z})$, $w_p(h_0) = k$. Then $w_{p^2}(f) = k$, $f \not\equiv 0 \pmod{p}$, yet $pf = pE_{p-1}^p f_0 \in M_{k-p(p-1)}(\mathbb{Z}/p^2\mathbb{Z})$.

The non-freeness over $\mathbb{Z}/p^2\mathbb{Z}$ of these quotient spaces, as well as the complicated nature of the weight filtration under θ modulo p^2 seems to prevent any kind of straightforward generalization of Jochnowitz-type arguments to the mod p^2 situation, as perhaps might be expected.

4. An example

As an illustration of Theorem 5 and specifically of Corollary 1, consider $\Delta \in M_{12}(\mathbb{Z}_p)$ and $p = 5$. We have $w_{p^2}(\Delta) = w_p(\Delta) = 12$, and further that $E_{p-1} = E_4 = Q$ and $E_{p+1} = E_6 = R$, and $\Delta = \frac{1}{1728}(Q^3 - R^2)$.

Hence, $A = X, C = Y$.

Moveover, $G_{22} = \alpha E_4^4 E_6 + \beta E_4 E_6^3$, where $\alpha = -19061/1841$, and $\beta = -5125/138$.

Since $\partial Q = -4R, \partial R = -6Q^2$, we have that

$$\begin{aligned}\partial\Delta &= \frac{1}{1728}(3Q^2\partial Q - 2R\partial R)\\ &= \frac{1}{1728}(3Q^2\cdot(-4R) - 2R\cdot(-6Q^2))\\ &= 0.\end{aligned}$$

We have

$$G_2 \equiv G_{22}E_4^5 + 5G_6^5E_4^3 \pmod{25},$$

using Proposition 1. Hence,

$$\begin{aligned}\theta\Delta &= -24G_2\Delta + \frac{1}{12}\partial\Delta\\ &= -24G_2\Delta\\ &\equiv (G_{22}E_4^5 + 5G_6^5E_4^3)\Delta \pmod{25}\end{aligned}$$

which displays $\theta\Delta$ as congruent modulo 25 to a form in $M_{54}(\mathbb{Z}_5)$.

The polynomial in $(\mathbb{Z}/25)\mathbb{Z}[X,Y]$ which corresponds to this form is

$$H(X,Y) = ((\alpha X^4Y + \beta XY^3)X^5 + 5(-504Y)^5X^3)\frac{X^3 - Y^2}{1728}.$$

As the polynomial $H(X,Y)$ is not divisible by X^5 in $(\mathbb{Z}/25)\mathbb{Z}[X,Y]$, there is no form h in $M_{34}(\mathbb{Z}_5)$ with $\theta\Delta \equiv h \pmod{25}$. Thus,

$$w_{p^2}(\theta\Delta) = 54 = 12 + 2 + 2\cdot p(p-1).$$

References

[1] I. Chen, I. Kiming, G. Wiese: 'On modular Galois representations modulo prime powers', to appear Int. J. Number Theory (2013).

[2] I. Chen, I. Kiming, J. B. Rasmussen: 'On congruences mod $\mathfrak{p}^m$ between eigenforms and their attached Galois representations', J. Number Theory (3) **130** (2010), 608–619.

[3] B. Edixhoven: 'The weight in Serre's conjectures on modular forms', Invent. Math. (3) **109** (1992), 563–594.

[4] N. Jochnowitz: 'Congruences between systems of eigenvalues of modular forms', Trans. Amer. Math. Soc. **270** (1982), 269–285.

[5] N. M. Katz: 'Higher congruences between modular forms', Ann. of Math. (2) **101** (1975), 332–367.

[6] N. M. Katz: 'A result on modular forms in characteristic p', in J.-P. Serre, D. B. Zagier (eds.): Modular functions of one variable V, Lecture Notes in Math. **601** (1977), 53–61.

[7] S. Lang: 'Introduction to modular forms', Springer 1976.

[8] J.-P. Serre: 'Congruences et formes modulaires [d'après H. P. F. Swinnerton-Dyer]', Séminaire Bourbaki, Lecture Notes in Math. **317**, 319–338, Springer 1973.

[9] J.-P. Serre: 'Formes modulaires et fonctions zêta p-adiques, in W. Kuyk, J.-P. Serre: Modular functions of one variable III, Lecture Notes in Math. **350**, Springer 1973.

[10] L. C. Washington: 'Introduction to cyclotomic fields', Grad. Texts in Math. **83**, Springer 1982.

(Imin Chen) Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, B.C., V5A 1S6, Canada

E-mail address: ichen@math.sfu.ca

(Ian Kiming) Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark

E-mail address: kiming@math.ku.dk